

Potts model with $q = 3$ and 4 states on directed Small-World network

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Monte Carlo simulations are performed to study the two-dimensional Potts models with $q = 3$ and 4 states on directed Small-World network. The disordered system is simulated applying the Heat bath Monte Carlo update algorithm. A first-order and second-order phase transition is found for $q = 3$ depending on the rewiring probability p , but for $q = 4$ the system presents only a first-order phase transition for any value p . This critical behavior is different from the Potts model on a square lattice, where the second-order phase transition is present for $q \leq 4$ and a first-order phase transition is present for $q > 4$.

Keywords: Monte Carlo simulation, spins, networks, Ising, Potts.

It was conjectured by Harris [1] that the sign of the critical exponent of the specific heat α determines whether spin systems are affected or not by randomness. For positive values of α the system with randomness or impurities has a critical behavior different from the pure system case. For negative values of α , on the other hand, the critical behavior of the system should be the same for both pure and impure cases. In particular, for two-dimensional regular lattices, the ferromagnetic Potts model with q states displays first order phase transitions for $q > 4$ [2, 3], while the pure ferromagnetic three-state Potts model has $\alpha = 1/3$, hence, according to the above-mentioned criterion we expect to find a different behavior for a random interaction system. However, Picco [4] and Lima et al. [5–8] studied this model with different type of disorder and did not find any relevant difference from the pure case.

The q -state Potts model has been studied in scale-free networks by Igloi and Turban [9] and depending on the value of q and of the degree-exponent γ first- and second-order phase transitions were found. This model was also studied by Lima [10] on *directed* Barabási-Albert(BA) networks, where only first-order phase transition has been obtained for any q -values with connectivity $z = 2$ and $z = 7$ of the *directed* BA network. Here, we studied the Potts model with $q = 3$ and 4 states. We also calculate the critical exponents ratio β/ν and γ/ν for second-order phase transitions that appears due to the SW disorder.

We consider the ferromagnetic Potts model with $q = 3$ and $q = 4$, on *directed* small-world networks where every site of a *directed* small-world network of size $N = L \times L$ have spin variables σ taking values 1, 2, 3 and 1, 2, 3, 4 for $q = 3$ and 4, respectively. With L being the side of a square lattice. In this network, created by Sánchez et al. [11] (see Fig. 1), we start from a two-dimensional square lattice consisting of sites linked to their four nearest neighbors by both outgoing and incomplete links. Then, with probability p , we reconnect nearest-neighbors

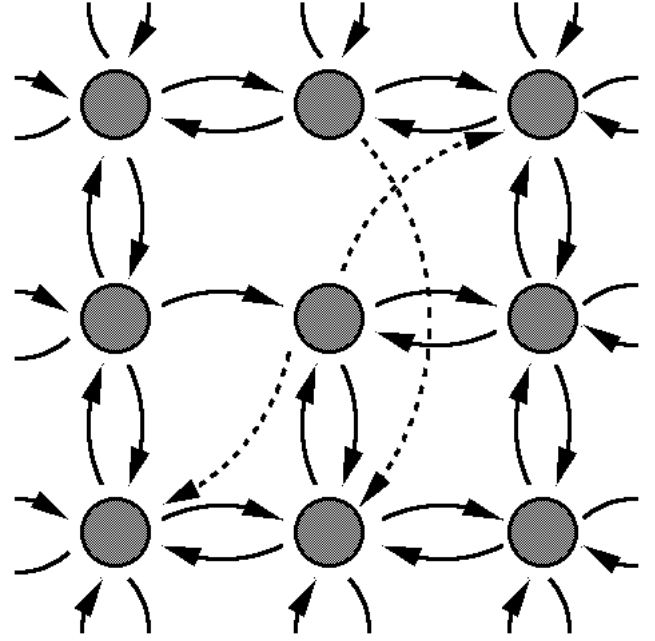


FIG. 1. Sketch of a *directed* small-world networks constructed from a square regular lattice in $d = 2$. Figure gently yielded by Juan M. Lopez from Sánchez et al. [11].

outgoing links to a different site chosen randomly. After repeating this process for every link, we are left with a network with a density p of SW *directed* links. Therefore, with this procedure every site will have exactly four outgoing links and different (random) number of incoming links. The time evolution of this system is given by a single spin-flip like dynamics with a probability p_i :

$$p_i = \frac{1}{[1 + \exp(2E_i/k_B T)]}. \quad (1)$$

The Hamiltonian of a q -states ferromagnetic Potts

model can be written as

$$H = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j}, \quad (2)$$

where δ is the Kronecker delta function, and the sum runs over all neighbors of σ_i .

The simulations have been performed applying the HeatBath update algorithm on different lattice sizes: $N = 64, 256, 1024, 4096$, and 16384 . For each system size quenched averages over the connectivity disorder are approximated by averaging over $R = 40$ independent realizations. For each simulation we have started with a uniform configuration of spins (the results are independent of the initial configuration). We ran 4×10^4 Monte Carlo steps (MCS) per spin with 2×10^4 configurations discarded for thermalization.

In studying the critical behavior of the model using the HeatBath algorithm we define the variable $e = E/N$, where E is the energy of system, and the magnetisation of system $M = (q \cdot \max[n_i] - N)/(q - 1)$, where $n_i \leq N$ denote the number of spins with ‘orientation’ $i = 1, \dots, q$. From the fluctuations of e measurements we can compute: the average of e , the specific heat C and the fourth-order cumulant of e ,

$$u(T) = [\langle E \rangle]_{av}/N, \quad (3)$$

$$C(T) = K^2 N [\langle e^2 \rangle - \langle e \rangle^2]_{av}, \quad (4)$$

$$B(T) = \left[1 - \frac{\langle e^4 \rangle}{3 \langle e^2 \rangle^2} \right]_{av}, \quad (5)$$

the temperature can be defined as $T = J/k_B K$, where k_B is the Boltzmann constant. Similarly, we can derive from the magnetization measurements the average magnetization ($m = M/N$), the susceptibility, and the magnetic cumulants,

$$m(T) = [\langle |m| \rangle]_{av}, \quad (6)$$

$$\chi(T) = K N [\langle m^2 \rangle - \langle |m| \rangle^2]_{av}, \quad (7)$$

$$U_4(T) = \left[1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2} \right]_{av}. \quad (8)$$

where in all the above equations $\langle \dots \rangle$ stands for a thermodynamic average and $[\dots]_{av}$ for an average over the 40 realizations.

To verify the transition order for this model, we apply finite-size scaling (FSS) [12]. Initially we search for the minima of the fourth-order parameter of Eq. (5). This quantity gives a qualitative as well as a quantitative description of the order of the transition [13]. It is known [14] that this parameter takes a minima value

B_{\min} at effective transition temperature $T_c(N)$. One can show [15] that for a second-order transition $\lim_{N \rightarrow \infty} (2/3 - B_{\min}) = 0$, even at T_c , while at a first-order transition the same limit measuring the same quantity is small and $(2/3 - B_{\min}) \neq 0$.

A more quantitative analysis can be carried out through the FSS of the C fluctuation C_{\max} , the susceptibility maxima χ_{\max} and the minima of the Binder parameter B_{\min} .

If the hypothesis of a first-order phase transition is correct, we should then expect, for large systems sizes, an asymptotic FSS behavior of the form [16–18],

$$C_{\max} = a_C + b_C N + \dots, \quad (9)$$

$$\chi_{\max} = a_\chi + b_\chi N + \dots, \quad (10)$$

$$B_{\min} = a_B + b_B/N + \dots, \quad (11)$$

if the hypothesis of a second-order phase transition is correct, we should then expect, for large systems sizes, an asymptotic FSS behavior of the form

$$C = C_{reg} + L^{\alpha/\nu} f_C(x)[1 + \dots], \quad (12)$$

$$m = L^{-\beta/\nu} f_m(x)[1 + \dots], \quad (13)$$

$$\chi = L^{\gamma/\nu} f_\chi(x)[1 + \dots], \quad (14)$$

$$\frac{dU_4}{dT} = L^{1/\nu} f_U(x)[1 + \dots], \quad (15)$$

where C_{reg} is a regular background term, ν , α , β , and γ are the usual critical exponents, and $f_i(x)$ are FSS functions with

$$x = (T - T_c) L^{1/\nu}, \quad (16)$$

being the scaling variable, and the brackets $[1 + \dots]$ indicate corrections-to-scaling terms. Therefore, from the size dependence of M and χ we obtain the exponents β/ν and γ/ν , respectively. The maxima value of susceptibility also scales as $L^{\gamma/\nu}$.

For each value of q , we apply the finite size scaling technique [12], and the same procedure is done for systems with different number of sites $N = 64, 256, 1024, 4096$, and 16384 . The critical temperature for infinite size system is estimated by using the fourth-order magnetization (Binder) cumulant.

In Fig. 2, we show the dependence of the energy u and magnetization m on the temperature T , obtained from simulations on directed SW network with lattice size $L = 8, 16, 32, 64$, and 128 and the rewiring probability $p = 0.0, 0.1$, and 0.9 . The shape of $m(T)$ and energy u curve, for

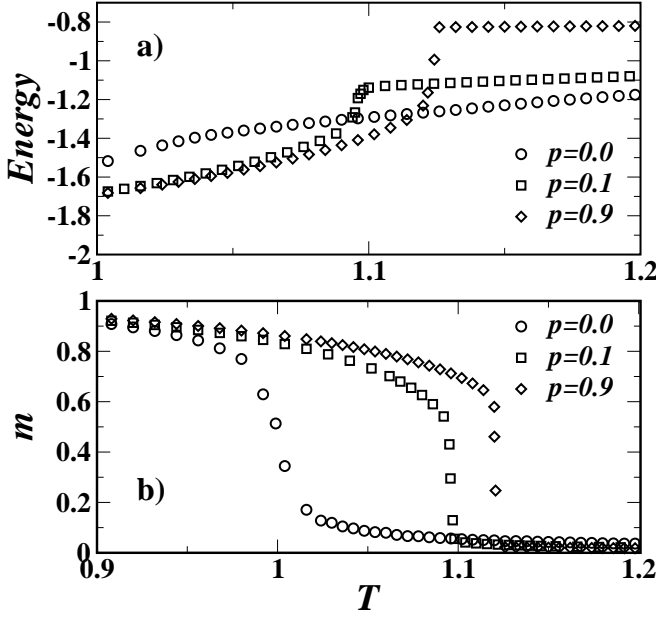


FIG. 2. Display of the energy (top panel) and magnetisation (bottom panel) against temperature T for $p = 0.0$ (circle), $p = 0.1$ (square), and $p = 0.9$ (diamond). Here $L=128$ and we are considering the case when $q = 3$.

the particular parameters used ($N = 16384$ and $q = 3$), suggests the existence of a second-order phase transition in the system for $p = 0.0$ and $p = 0.1$, and a first-order phase transition in the system for $p = 0.9$. The phase transition occurs at the value of the critical parameter T_c . The energetic Binder cumulant as a function of the reduced temperature T is shown in Fig. 3 for $p = 0.1$ and 0.9 and different lattice sizes ($L = 8$ to 128). From the figure one can see a typical second-order phase transition (for a large system $B_e(T) \rightarrow 2/3$) and a first-order phase transition is observed for $p = 0.1$ and 0.9 , respectively.

In Fig. 4, the difference $2/3 - B_{min}$ is shown as a function of the parameter $1/\sqrt{N}$ for $p = 0.1$ and $p = 0.9$. For $p = 0.1$, a second-order transition takes place since the $\lim_{N \rightarrow \infty} (2/3 - B_{i,min}) = 0$, even at T_c . However, for $p = 0.9$ a first-order transition is observed, because one has $(2/3 - B_{i,min}) \neq 0$.

We display the scalings for natural logarithm for the dependence of the magnetization m on inflection point at $K = T_c(L)$ and $p = 0.1$ for $q = 3$ in the Figure 5. The slopes of curves correspond to the exponent ratio β/ν according to Eq. 13. The obtained exponents are $\beta/\nu = 0.24(5)$. The exponents ratio γ/ν are obtained from the slopes of the straight lines with $\gamma/\nu = 1.5(1)$ for SW, as presented in Fig. 6 and obtained from Eq. 14. The results present a reliable indication in favor of the Harris criterium, error bars are only statistical, and much larger systems might give different exponents, also, the exponents ratio β/ν and γ/ν obey the hyper-scaling law $\gamma/\nu + 2\beta/\nu = d$.

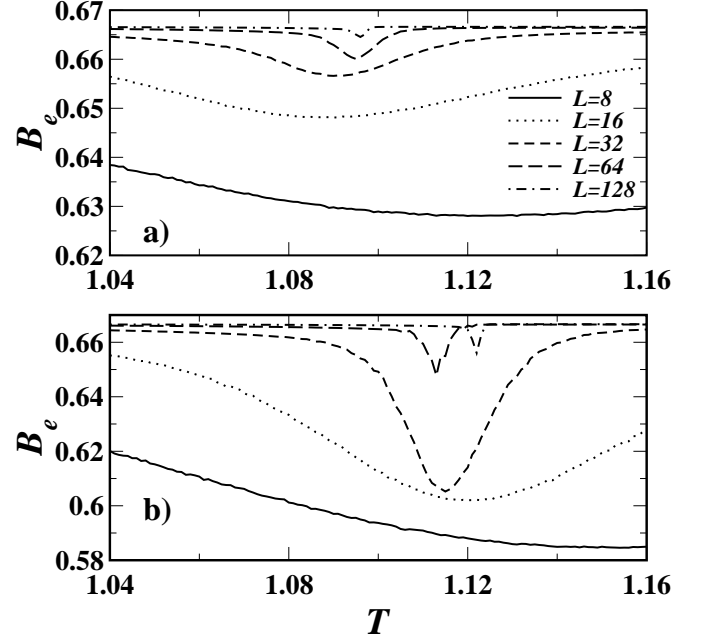


FIG. 3. Plot $B_e(T)$ versus T for: a) $p = 0.1$ and b) $p = 0.9$ for different size lattices $L = 8$ solid line, $L = 16$ dotted line, $L = 32$ dashed line, $L = 64$ long dashed line, and $L = 128$ dotted-dashed line. In all cases we used $q = 3$.

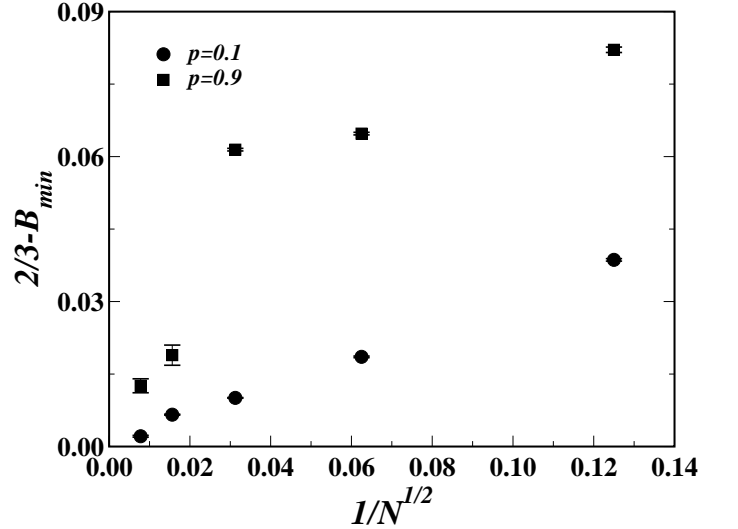


FIG. 4. Plot of $2/3 - B_{min}$ at T_c as a function of $1/\sqrt{N}$ for several values of the system size $N = 64$ to $16,384$ sites for $p = 0.1$ (circles) and $p = 0.9$ (squares).

Next, we study the case where $q = 4$. In Fig. 7, as in the Fig. 2, we show the dependence of the magnetization m and energy u on the temperature T , obtained from simulations on directed with lattice size $L = 8, 16, 32, 64$, and 128 with $(L \times L = N)$ sites and the rewiring probability $p = 0.0$, $p = 0.1$, and $p = 0.9$. The shape of $m(T)$ and energy u curve, for a given value of $N = 16384$ sites and $q = 4$, suggests the presents of the second-order phase

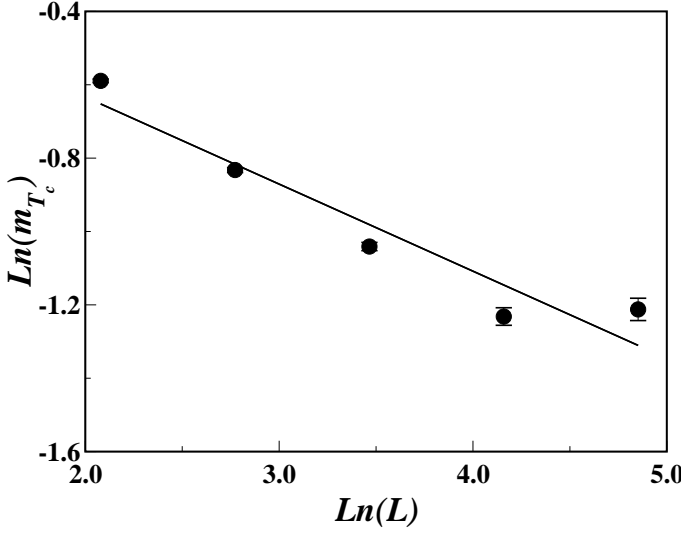


FIG. 5. Display of the magnetisation at the inflection point versus the size system L $p = 0.1$, $q = 3$.

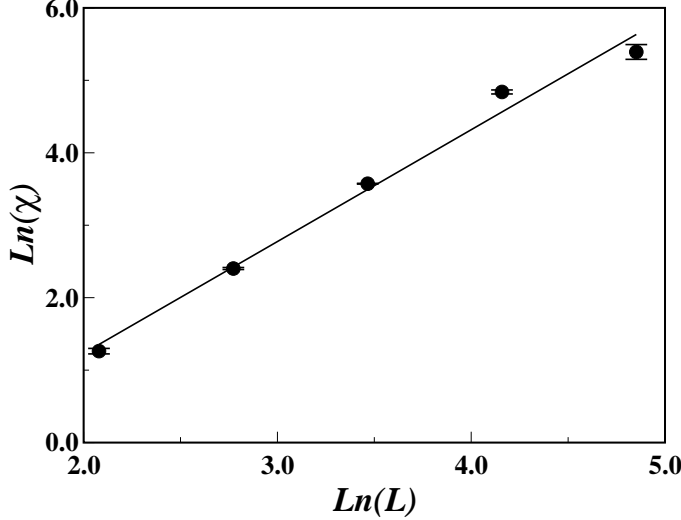


FIG. 6. Logarithmic plots of the susceptibility at T_c versus the size system L for $p = 0.1$, $q = 3$.

transition in the system for $p = 0.0$, but also suggests the presents of the first-order phase transition in the system for $p = 0.1$ and 0.9 .

In Fig. 8, as in the Fig. 5, we plot the difference $2/3 - B_{min}$ as a function of the parameter $1/\sqrt{N}$ for different probabilities $p = 0.1$ and $p = 0.9$. Unlike the $q = 3$ case, for both values $p = 0.1$ and 0.9 a first-order transition is observed, because $(2/3 - B_{i,min}) \neq 0$.

In conclusion, we have presented simulations for Potts model with $q = 3$, and 4 states on directed SW network. The disordered system is simulated applying the Heat-Bath Monte Carlo update algorithm. The Potts model with $q = 3$ does display a second-order phase for rewiring probability $p = 0.1$, with exponent ratio $\beta/\nu = 0.24(5)$

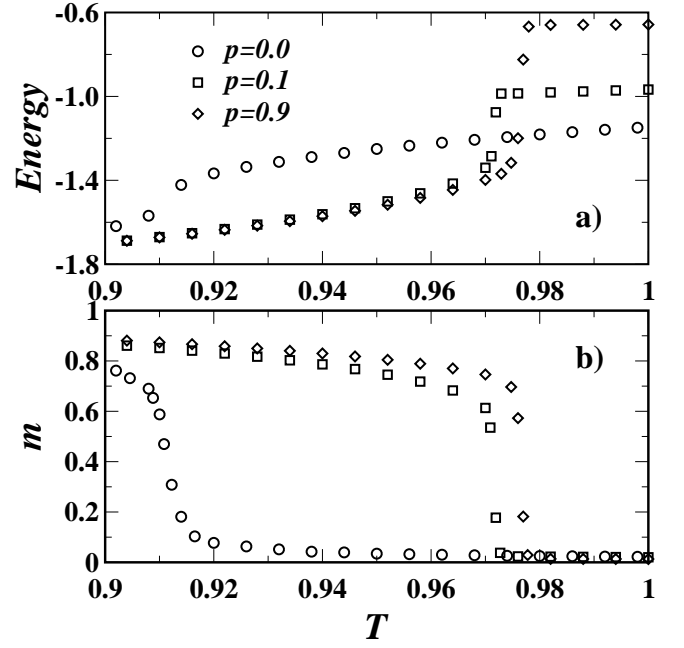


FIG. 7. The same plot of Fig. 2, but now for $q = 4$.

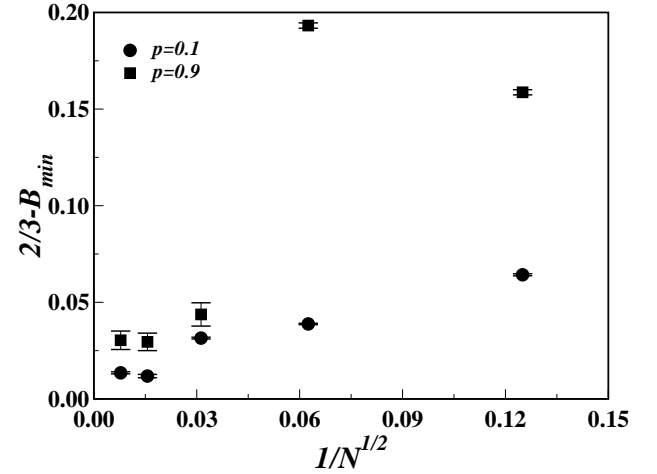


FIG. 8. The same plot of Fig. 5, but now for $q = 4$.

and $\gamma/\nu = 1.5(1)$ that are different of the Potts model on a regular lattice, where, the specific-heat exponent $\alpha = 2/3$ is a good candidate for a change of the critical exponents, that agree with the Harris criterium [1] and obey the hyper-scaling law $\gamma/\nu + 2\beta/\nu = d$ and for case of $p = 0.9$ we have a first-order phase transition. In the case $q = 4$ both values here studied rewiring probability $p = 0.1$ and 0.9 present a first-order phase transition as showed in the Fig. 7 and 8, that again agree with Harris criteria. In summary, the behavior of Potts model for $q = 3$ and 4 , here studied, is due to the directed links of the SW networks, where can have short and long range interaction.

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- [1] A.B. Harris, J. Phys. C **7**, 1671 (1974).
- [2] F.Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
- [3] C. Tsallis, Phys. Rep. **268**, 305 (1996).
- [4] M. Picco, Phys. Rev. B **54**, 1493(1996).
- [5] F.W.S. Lima, U.M.S. Costa, M.P. Almeida, and J.S. Andrade Jr., Eur. Phys. J. B **17**, 111 (2000).
- [6] F. W. S. Lima, U. L. Fulco, and R. N. Costa Filho, Phys. Rev. E **71**, 036105 (2005).
- [7] F. W. S. Lima, R. N. Costa Filho, and U. M. S. Costa, J. Mag. Mag. Mat. **270**, 182 (2004)
- [8] F. W. S. Lima, U. M. S. Costa, and R. N. Costa Filho, Physica A **387**, 1545 (2008).
- [9] F. Igloi and L. Turban, Phys. Rev. E **66**, 036140 (2002), cond-mat/0206522.
- [10] F. W. S. Lima, Commun. Comput. Phys. **2**, 358-366 (2007).
- [11] Alejandro D. Sanchez, Juan M. Lopez, and Miguel A. Rodriguez, Phys. Rev. Lett. **88**, 048701-1 (2002).
- [12] See Finite Size Scaling and Numerical Simulation of Statistical Systems, edited by V. Privman (World Scientific, Singapore, 1990).
- [13] M.S.S. Challa, D. P. Landau, K. Binder, Phys. Rev. B, **34**, 1841 (1986).
- [14] W. Janke, Phys. Rev. B **47**, 14757 (1993).
- [15] K. Binder, D. J. Herrmann, Monte-Carlo Simulation in Statistical Phys., (Springer-Verlag, Berlin, 1988), p. 61-62.
- [16] W. Janke, Mohammad Katoot and R. Villanova, Phys. Rev. B, **49**, 9644 (1994).
- [17] W. Janke, R. Villanova, Phys. Lett. A **209**, 179 (1995).
- [18] F.W.S. lima, J.E. Moreira, J.S. Andrade Jr., and U.M.S. Costa, Eur. Phys. J. B **13**, 107 (2000).